

After the Explosion: Dirichlet Forms and Boundary Problems for Infinite Graphs

Robert Carlson
Department of Mathematics
University of Colorado at Colorado Springs
carlson@math.uccs.edu

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Abstract

Formal Laplace operators are analyzed for a large class of resistance networks with vertex weights. The graphs are completed with respect to the minimal resistance path metric. Compactness and a novel connectivity hypothesis for the completed graphs play an essential role. A version of the Dirichlet problem is solved. Self adjoint Laplace operators and the probability semigroups they generate are constructed using reflecting and absorbing conditions on subsets of the graph boundary.

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1 Introduction

This work has its roots in the challenge of extending differential equation models for diffusion or wave propagation from domains in Euclidean space to infinite graphs intended to resemble biological transport systems such as the arteries of the human circulatory system. Such biological systems can include enormous numbers of branching segments. Short time transport across the network is essential, so treelike structures with small numbers of large edges and vast collections of microscopic edges are typical. Faced with such complex heterogeneous structures, one hopes that appropriate infinite graph models will suggest useful structural features and robustly posed problems.

Building on an earlier 'quantum graph' analysis of such problems [1], this work uses infinite graph and operator theoretic methods to treat a class of continuous time Markov chains. Recall that continuous time Markov chains use a system of constant coefficient differential equations

$$\frac{dP}{dt} = QP, \quad P(0) = I. \quad (1.1)$$

to describe the evolution of probability densities $X(t) = X(0)P(t)$ on a finite or countably infinite set of states. An associated graph may be constructed by connecting states (vertices) i and j with an edge if $Q_{ij} \neq 0$.

In the finite state case the solution of (1.1) is simply $P(t) = e^{Qt}$. When the set of states is infinite the formal description of the operator Q may not be adequate to determine the semigroup e^{Qt} , an issue known in probability as the problem of explosions. Infinite graph models inspired by biological transport systems will typically face the explosion problem. By imposing restrictions on both the form of the Markov chain generator Q and the structure of the associated graph viewed as a metric space, this work provides a resolution in terms of 'reflecting' and 'absorbing' behavior at a graph boundary.

It will be advantageous to use the Dirichlet form theory [4, 9]. To that end, consider a graph \mathcal{G} whose edges $e = [u, v]$ are equipped with positive weights $R(u, v)$ which are interpreted as edge length. With $C(u, v) = 1/R(u, v)$, a symmetric bilinear form for functions on the vertex set is defined by

$$B(f, g) = \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{u \sim v} C(u, v) (f(v) - f(u))(g(v) - g(u)). \quad (1.2)$$

Each vertex is also given a positive weight $\mu(v)$. Formal semigroup generators

Δ_μ are defined by

$$\Delta_\mu f(v) = \frac{1}{\mu(v)} \sum_{u \sim v} C(u, v)(f(v) - f(u)). \quad (1.3)$$

$R(u, v)$ is often interpreted as electrical resistance. The electrical network analogy is treated at length in [6, 18]. The recent work [10] treats electrical currents in a context similar to this paper, while [11] treats related topological questions. An analysis of function theory on infinite trees motivated by modeling the human lungs is in [19]. With the domain of functions with finite support, Δ_μ is a symmetric operator on $l^2(\mu)$. In contrast to this paper, other recent work [3, 13, 15, 16] has stressed cases when this symmetric operator is essentially selfadjoint, and so behavior at the graph boundary is not an issue.

The vertex set \mathcal{V} of an edge weighted locally finite graph \mathcal{G} can be equipped with a metric $d(u, v)$ obtained by minimizing the sum of the edge lengths of paths from u to v . By completing this metric space we obtain a metric space $\overline{\mathcal{G}}$ in which one can discuss features like the graph boundary and compactness. If \mathcal{G} is a tree, then distinct points of $\overline{\mathcal{G}}$ can be separated by deleting a suitable edge. Generalizing this idea, our graphs will be required to have 'weakly connected' completions, with the property that, for any two distinct points, any path joining them must include an edge from a finite set. This generalization identifies a rich class of edge weighted graphs with useful topological and function theoretic properties.

The properties of weakly connected graph completions are developed in the second section. In addition to trees, arbitrary graphs with finite volume have weakly connected completions. This class is also preserved if we add suitably constrained edge sequences to a graph. Weakly connected completions are totally disconnected metric spaces. When also compact, these spaces are topologically stable with respect to decrease of the metric. The weakly connected class will be characterized using the separation of points property for an algebra of 'eventually flat' functions.

The third section treats the bilinear forms, vertex weights, and associated operators. The choice of vertex weights typically used for discrete time Markov chains are contrasted with weights making Δ_μ resemble a discretized second derivative. The bilinear form is used to construct several 'Sobolev style' Hilbert spaces on \mathcal{G} whose elements extend continuously to $\overline{\mathcal{G}}$.

Two main problems are treated in the fourth section. The first, a version of the Dirichlet problem, asks for conditions under which continuous func-

tions on $\partial\overline{\mathcal{G}}$ have a unique harmonic extension to \mathcal{G} . An example shows that a lack of compactness can lead to a negative result. Using assumptions of compactness and weak connectivity, a general positive result is established. The second problem is the resolution of the explosion problem in terms of reflecting and absorbing boundary conditions. The semigroups generated by the operators defined using these boundary conditions are positivity preserving contractions on $l^1(\mu)$.

Despite the connections with probability, this work will not explicitly use probabilistic techniques or interpretations. We simply mention the classic work [7], and the recent works [9, 17, 21] as pointers to the enormous literature related to analysis of infinite state Markov chains.

2 Weakly connected graphs

2.1 Topology

\mathcal{G} will denote a simple graph with a countable vertex set \mathcal{V} and a countable edge set \mathcal{E} . Each vertex will have at least one and at most finitely many incident edges. Vertices of degree 1 are *boundary vertices*; the rest are *interior vertices*. \mathcal{G} is assumed to have *edge weights (resistances)*. That is, there is a function $R : \mathcal{E} \rightarrow (0, \infty)$, denoted by $R(u, v)$ when $[u, v] \in \mathcal{E}$. General references on graphs are [2, 5]. Edge weights, considered the length of the edges, are commonly identified with electrical network resistance [6] or [18], and then *edge conductance* is the reciprocal $C(u, v) = 1/R(u, v)$ if $R(u, v) > 0$, and 0 otherwise.

A *finite path* (sometimes called a walk) in \mathcal{G} connecting vertices u and v is a finite sequence of vertices $u = v_0, v_1, \dots, v_K = v$ such that $[v_k, v_{k+1}] \in \mathcal{E}$ for $k = 0, \dots, K-1$. \mathcal{G} is connected there is a finite path from u to v for all $u, v \in \mathcal{V}$. Define a metric on (the vertices of) \mathcal{G} by

$$d(u, v) = \inf_{\gamma} \sum_k R(v_{k+1}, v_k), \quad (2.1)$$

the infimum taken over all finite paths γ joining u and v . If there is no finite path from u to v then $d(u, v) = \infty$. $\overline{\mathcal{G}}$, with the extended metric d , will denote the metric space completion [20, p. 147] of \mathcal{G} .

Extending the combinatorial notion of path, a *path* in $\overline{\mathcal{G}}$ will be a sequence $\{v_k\}$ with $v_k \in \mathcal{V}$, $[v_k, v_{k+1}] \in \mathcal{E}$, where the index set may be finite (finite

path), the positive integers (a ray), or the integers (a double ray). The role of continuous paths in $\overline{\mathcal{G}}$ is played by paths going from $u \in \overline{\mathcal{G}}$ to $v \in \overline{\mathcal{G}}$, which in the double ray case requires $\lim_{k \rightarrow -\infty} d(v_k, u) = 0$ and $\lim_{k \rightarrow \infty} d(v_k, v) = 0$. The ray case is similar. A path for which all vertices are distinct is a *simple path*. If \mathcal{G} is connected then there is a path joining any pair of points $u, v \in \overline{\mathcal{G}}$. Modifying ideas from [1], say that $\overline{\mathcal{G}}$ is *weakly connected* if for every pair of distinct points $u, v \in \overline{\mathcal{G}}$ there is a finite set W of edges in \mathcal{G} such that every path from u to v contains an edge from W .

One may extend \mathcal{G} to a metric graph \mathcal{G}_m by identifying the combinatorial edge $[u, v]$ with an interval of length $R(u, v)$. With the usual metric on \mathcal{G}_m , its vertex set will be isomorphic to \mathcal{G} . By this device some of the results of [1], which should be consulted for more details, carry over to the present context. The next result is a simple example.

Proposition 2.1. *If \mathcal{T} is a tree then $\overline{\mathcal{T}}$ is weakly connected.*

The volume of a graph is defined as the sum of its edge lengths,

$$\text{vol}(\mathcal{G}) = \sum_{[v_1, v_2] \in \mathcal{E}} R(v_1, v_2).$$

Finite volume graphs also have weakly connected completions [1].

Proposition 2.2. *If $\text{vol}(\mathcal{G}) < \infty$ then $\overline{\mathcal{G}}$ is weakly connected.*

Proof. The main case considers distinct points x and y in $\overline{\mathcal{G}} \setminus \mathcal{G}$. Remove a finite set of edges from \mathcal{E} so that the remaining edgeset \mathcal{E}_1 satisfies

$$\sum_{[v_1, v_2] \in \mathcal{E}_1} R(v_1, v_2) < \frac{d(x, y)}{2}.$$

Proceeding with a proof by contradiction, suppose $(\dots, v_{-1}, v_0, v_1, \dots)$ is a path from x to y using only edges in \mathcal{E}_1 . Then for n sufficiently large $d(v_{-n}, v_n) > \frac{d(x, y)}{2}$, but there is a simple path from v_{-n} to v_n using only edges from \mathcal{E}_1 , so $d(v_{-n}, v_n) < \frac{d(x, y)}{2}$. \square

The next result gives conditions allowing edges to be added to a weakly connected graph without disturbing that property.

Theorem 2.3. *Suppose the graph \mathcal{G}_0 , with vertex set \mathcal{V} , has a weakly connected completion $\overline{\mathcal{G}}_0$. Using the same vertex set \mathcal{V} , enlarge \mathcal{G}_0 to a graph \mathcal{G}_1 by adding a sequence \mathcal{E}_1 of edges e_n whose lengths R_n satisfy $\lim_{n \rightarrow \infty} R_n = 0$. Assume there is a positive constant C such that*

$$d_{\mathcal{G}_1}(u, v) \leq d_{\mathcal{G}_0}(u, v) \leq C d_{\mathcal{G}_1}(u, v), \quad u, v \in \mathcal{V}.$$

Then $\overline{\mathcal{G}}_1$ is weakly connected.

Proof. First note that $\overline{\mathcal{G}}_1 \setminus \mathcal{G}_1 = \overline{\mathcal{G}}_0 \setminus \mathcal{G}_0$, since the set of Cauchy sequences of vertices has not changed. It will suffice to consider distinct points u and v in $\overline{\mathcal{G}}_1 \setminus \mathcal{G}_1$ which are joined by a path in $\overline{\mathcal{G}}_1$. Let W_0 be a finite set of edges in \mathcal{G}_0 such that every path in $\overline{\mathcal{G}}_0$ from u to v contains an edge $[v_1, v_2]$ from W_0 .

Pick $\epsilon > 0$ such that $\epsilon < R(v_1, v_2)$ for all edges $[v_1, v_2] \in W_0$. Find N so that the lengths R_n of edges $e_n \in \mathcal{E}_1$ satisfy $R_n < \epsilon/C$ for $n > N$. Let W_1 be the set of edges $W_0 \cup \{e_1, \dots, e_N\}$ in \mathcal{G}_1 . Suppose there is a path γ in $\overline{\mathcal{G}}_1$ joining u to v , but not containing any edge from W_1 .

Let Γ be the set of edges $e_j \in \mathcal{E}_1$ which are in γ . Γ is not empty since γ contains at least one edge not in \mathcal{G}_0 . Since γ contains no edge from W_1 , the edges $e_j \in \Gamma$ have length $R_j < \epsilon/C$. If an edge $e_j = [v_j, v_{j+1}] \in \Gamma$ has $d_{\mathcal{G}_0}(v_j, v_{j+1}) < \epsilon$, then e_j can be replaced by a finite path γ_j in \mathcal{G}_0 with length at most $C R_j$, and containing no edge from W_0 . Thus there is at least one $e_j = [v_j, v_{j+1}] \in \Gamma$ with $d_{\mathcal{G}_0}(v_j, v_{j+1}) \geq \epsilon$. But the inequalities $d_{\mathcal{G}_1}(v_j, v_{j+1}) < \epsilon/C$ while $d_{\mathcal{G}_0}(v_j, v_{j+1}) \geq \epsilon$ contradict the hypotheses, so no such path from u to v exists, and $\overline{\mathcal{G}}_1$ is weakly connected. \square

The conclusion of Theorem 2.3 may be false if the edge lengths R_n have a positive lower bound. Start with \mathcal{G}_0 which is weakly connected and connected. Take distinct points $x, y \in \overline{\mathcal{G}}_0 \setminus \mathcal{G}_0$. Suppose γ is a path from x to y . Take sequences of vertices x_n, y_n from γ with $x_n \rightarrow x$ and $y_n \rightarrow y$ such that $x_n \neq y_n$, and $[x_n, y_n]$ is not an edge in \mathcal{G}_0 . Form \mathcal{G}_1 by adding edges $[x_n, y_n]$ to \mathcal{G}_0 , with $R(x_n, y_n) = d_{\mathcal{G}_0}(x_n, y_n)$. Since $d_{\mathcal{G}_1}(u, v) = d_{\mathcal{G}_0}(u, v)$ for all vertices u, v , the vertex sets for \mathcal{G}_0 and \mathcal{G}_1 are isometric. The edge lengths $R(x_n, y_n)$ have a positive lower bound, and $\overline{\mathcal{G}}_1$ is not weakly connected,

Theorem 2.4. *Assume $\overline{\mathcal{G}}$ is weakly connected. If U and V are disjoint compact subsets of $\overline{\mathcal{G}}$, then there is a finite set W of edges in \mathcal{G} such that every path from U to V contains an edge in W .*

Proof. Since $\overline{\mathcal{G}}$ is weakly connected, if $u \in U$ and $v \in V$ there is a finite set $W(u, v)$ of edges in \mathcal{G} such that every path from u to v contains an edge in W . Take $\epsilon > 0$ such that $\epsilon < R(v_1, v_2)$ for all edges $[v_1, v_2] \in W$. If $z_1 \in B_\epsilon(u)$, the open ϵ ball centered at u , and $z_2 \in B_\epsilon(v)$, then every path from z_1 to z_2 contains an edge in $W(u, v)$. The collection $\{B_\epsilon(u) \times B_\epsilon(v), u \in U, v \in V\}$ is an open cover of the compact set $U \times V$, so there is a finite subcover $B_{\epsilon_n}(u_n) \times B_{\epsilon_n}(v_n)$ for $n = 1, \dots, N$. Take $W = \cup_n W(u_n, v_n)$. \square

Suppose W is a nonempty finite set of edges in \mathcal{E} . For $x \in \overline{\mathcal{G}}$, let $U_W(x)$ be the set of points $y \in \overline{\mathcal{G}}$ which can be connected to x by a path containing no edge of W .

Lemma 2.5. *For all $x \in \overline{\mathcal{G}}$ the set $U_W(x)$ is both open and closed in $\overline{\mathcal{G}}$.*

Proof. Take $\epsilon > 0$ such that $\epsilon < R(v_1, v_2)$ for all edges $[v_1, v_2] \in W$. Suppose y and z are vertices with $d(z, y) < \epsilon$, so there is a path of length smaller than ϵ from y to z . If there is a path from y to x containing no edge from W , then by concatenating these paths there is a path from z to x containing no edge from W . This shows that $U_W(x)$ and the complement of $U_W(x)$ are both open. \square

Theorem 2.6. *A weakly connected $\overline{\mathcal{G}}$ is totally disconnected.*

Proof. Suppose v_1 and v_2 are distinct points in $\overline{\mathcal{G}}$, with W being a finite set of edges in \mathcal{G} such that every path from v_1 to v_2 contains an edge from W . The set $U_W(v_1)$ is both open and closed. Since $U_W(v_1)$ and $U_W(v_2)$ are disjoint, $U_W(v_2) \subset U_W^c(v_1)$, the complement of $U_W(v_1)$ in $\overline{\mathcal{G}}$. Thus v_1 and v_2 lie in different connected components. \square

If $\overline{\mathcal{G}}$ is totally disconnected and compact, it has a rich collection of clopen sets, that is sets which are both open and closed. In fact [1] or [12, p. 97] for any $x \in \overline{\mathcal{G}}$ and any $\epsilon > 0$ there is a clopen set U such that $x \in U \subset B_\epsilon(x)$. In particular any compact subset of $\overline{\mathcal{G}}$ can then be approximated by a clopen set.

Changing the metric on \mathcal{G} may change the completion $\overline{\mathcal{G}}$ and the functions on \mathcal{G} that extend continuously to graphbar. Given \mathcal{G} and two weight functions R_0 and R_1 , let $\overline{\mathcal{G}}_0$ and $\overline{\mathcal{G}}_1$ denote the completions of \mathcal{G} with respect to the associated metrics d_0 and d_1 . Say that the weight function $R_1(u, v)$ is smaller than the weight function $R_0(u, v)$ if $R_1(u, v) \leq R_0(u, v)$ for every edge $[u, v] \in \mathcal{E}$. If R_1 is smaller than R_0 , the metric d_1 on \mathcal{G} extends to a pseudometric

[20, p. 140-141] on $\overline{\mathcal{G}}_0$; that is, there may be distinct points $x, y \in \overline{\mathcal{G}}_0$ with $d_1(x, y) = 0$. The next result establishes a stability property for weakly connected graph completions

Theorem 2.7. *Suppose $\overline{\mathcal{G}}_0$ is weakly connected. If R_1 is smaller than R_0 then the pseudometric on $\overline{\mathcal{G}}_0$ induced by R_1 is a metric. If in addition $(\overline{\mathcal{G}}_0, d_0)$ is compact, then $(\overline{\mathcal{G}}_0, d_0)$ and $(\overline{\mathcal{G}}_0, d_1)$ are homeomorphic.*

Proof. The usual construction [20, p. 147] identifies the completion of a metric space with equivalence classes of Cauchy sequences, two sequences $\{x_n\}$ and $\{y_n\}$ being equivalent when $d(x_n, y_n) \rightarrow 0$. Since \mathcal{G} is locally finite, it is only possible to have distinct points $x, y \in \overline{\mathcal{G}}_0$ with $d_1(x, y) = 0$ if $x, y \in \overline{\mathcal{G}}_0 \setminus \mathcal{G}$.

Suppose x and y are distinct points of $\overline{\mathcal{G}}_0 \setminus \mathcal{G}$, and that W is a finite set of edges such that any path from x to y contains an edge from W . Let $\{x_n\}$ and $\{y_n\}$ be sequences in \mathcal{V} with $x_n \rightarrow x$ and $y_n \rightarrow y$. By Lemma 2.5 there is an N such that $n \geq N$ implies any path in \mathcal{G} from x_n to y_n must contain an edge from W . Let $0 < \epsilon < R_1(u, v)$ for all edges $[u, v] \in W$. Then $d_1(x_n, y_n) \geq \epsilon$ for all $n \geq N$, so the sequences $\{x_n\}$ and $\{y_n\}$ are still inequivalent Cauchy sequences with respect to d_1 , showing that d_1 is a metric on $\overline{\mathcal{G}}_0$.

Let ι denote the identity map from $(\overline{\mathcal{G}}_0, d_0)$ to $(\overline{\mathcal{G}}_0, d_1)$, which is distance reducing, so continuous. If $(\overline{\mathcal{G}}_0, d_0)$ is compact, then so is $(\overline{\mathcal{G}}_0, d_1)$. Following [8, p.123], suppose $K \subset (\overline{\mathcal{G}}_0, d_0)$ is closed. Then $(\iota^{-1})^{-1}(K) = \iota(K)$ is compact, so closed, and ι^{-1} is continuous. \square

2.2 Function theory

There is an algebra of functions with pointwise addition and multiplication well matched to the weakly connected graph completions. Define the 'eventually flat' functions \mathbb{A} to be the real algebra of functions $\phi : \mathcal{V} \rightarrow \mathbb{R}$ such that the set of edges $[u, v]$ in \mathcal{E} with $\phi(u) \neq \phi(v)$ is finite.

Lemma 2.8. *All functions $\phi \in \mathbb{A}$ extend continuously to $\overline{\mathcal{G}}$.*

Proof. Let W be the finite set of edges $[u, v]$ with $\phi(u) \neq \phi(v)$. Suppose $x \in \overline{\mathcal{G}}$, $x_n \in \mathcal{G}$, and $x_n \rightarrow x$. If $x \in \mathcal{G}$ then $x_n = x$ for n sufficiently large. Suppose $x \in \overline{\mathcal{G}} \setminus \mathcal{G}$. By Lemma 2.5 the set $U(W)(x)$ is open, so there is an N such that $n \geq N$ implies $x_n \in U_W(x)$ and $\phi(x_n) = \phi(x_{n+1})$. Take $\phi(x) = \lim_{n \rightarrow \infty} \phi(x_n)$. \square

Making use of this lemma, functions $\phi \in \mathbb{A}$ are extended continuously to functions on $\overline{\mathcal{G}}$.

Lemma 2.9. *If \mathcal{G} is connected, then any $\phi \in \mathbb{A}$ has finite range. For $c \in \mathbb{R}$, $\phi^{-1}(c)$ is a clopen set in $\overline{\mathcal{G}}$.*

Proof. Suppose $\phi \in \mathbb{A}$ is not constant, and suppose $u \in \mathcal{V}$. Find a path $(u = v_0, v_1, \dots, v_N)$ such that $\phi(v_n) = \phi(u)$ for $n \leq N$ and $\phi(v_N) \neq \phi(w)$ for some w adjacent to v_N . Since the set of such vertices v_N is finite, $\phi(u)$ has one of a finite set of values.

For $c \in \mathbb{R}$, $\phi^{-1}(c)$ is a closed set, and its complement in $\overline{\mathcal{G}}$ is the union of a finite collection of closed sets. □

The next result shows that \mathbb{A} separates points of $\overline{\mathcal{G}}$ if and only if $\overline{\mathcal{G}}$ is weakly connected.

Theorem 2.10. *Suppose $\overline{\mathcal{G}}$ is weakly connected. If x and y are distinct points of $\overline{\mathcal{G}}$, there is a function $\phi \in \mathbb{A}$ whose range is $\{0, 1\}$ such that $\phi(z) = 0$ for z in an open neighborhood U of x , and $\phi(z) = 1$ for z in an open neighborhood V of y .*

Conversely, if \mathbb{A} separates points of $\overline{\mathcal{G}}$, then $\overline{\mathcal{G}}$ is weakly connected.

Proof. Let W be a finite set of edges in \mathcal{G} such that every path from x to y contains an edge from W . By Lemma 2.5 the set $U_W(x)$ is both open and closed in $\overline{\mathcal{G}}$, as is $U_W^c(x)$. Define $\phi(z) = 0$ for $z \in U_W(x)$ and $\phi(z) = 1$ for $z \in U_W^c(x)$. For every vertex v and adjacent vertex w we have $\phi(v) = \phi(w)$ unless $[v, w] \in W$. Since W is finite, $\phi \in \mathbb{A}$.

In the other direction, suppose \mathbb{A} separates points of $\overline{\mathcal{G}}$. Let x and y be distinct points in $\overline{\mathcal{G}}$, and suppose $\phi \in \mathbb{A}$ with $\phi(x) < \phi(y)$. Let W be the finite set of edges $[u, v]$ such that $\phi(u) \neq \phi(v)$. If γ is any path starting at x which contains no edge from W , then ϕ must be constant along γ . That is, every path from x to y must contain an edge from W , so $\overline{\mathcal{G}}$ is weakly connected. □

Corollary 2.11. *Suppose $\overline{\mathcal{G}}$ is weakly connected. If Ω and Ω_1 are nonempty disjoint compact subsets of $\overline{\mathcal{G}}$, then there is a function $f \in \mathbb{A}$ such that $0 \leq f \leq 1$,*

$$f(x) = 1, \quad x \in \Omega, \quad f(y) = 0, \quad y \in \Omega_1.$$

Proof. First fix $y \in \Omega_1$. Using Theorem 2.10, find a finite cover U_1, \dots, U_N of Ω by open sets with corresponding functions f_1, \dots, f_N which satisfy $f_n(z) = 1$ for all z in some open neighborhood V_y of y , while $f_n(z) = 0$ for all z in U_n .

Define $F_y = f_1 \cdots f_N$. Find a finite collection F_1, \dots, F_N whose corresponding open sets V_1, \dots, V_N cover Ω_1 . The function $f = (1 - F_1) \cdots (1 - F_N)$ has the desired properties. \square

The combination of Theorem 2.10 and the Stone-Weierstrass Theorem yields the next result.

Theorem 2.12. *If $\overline{\mathcal{G}}$ is weakly connected and compact, then \mathbb{A} is uniformly dense in the continuous functions on $\overline{\mathcal{G}}$.*

3 Quadratic forms

3.1 Weights and forms

Vertex weights and the corresponding measure are now added to the edge weighted graph \mathcal{G} . A vertex weight function $\mu : \mathcal{V} \rightarrow (0, \infty)$ provides the Borel measure $\mu(U) = \sum_{v \in U} \mu(v)$ on \mathcal{G} , which may be extended [20, p. 257] to $\overline{\mathcal{G}}$ by defining the measure of $\overline{\mathcal{G}} \setminus \mathcal{G}$ to be 0. The real Hilbert space $l^2(\mu)$ will consist of functions $f : \mathcal{V} \rightarrow \mathbb{R}$ with $\sum_{v \in \mathcal{V}} f(v)^2 \mu(v) < \infty$, and inner product $\langle f, g \rangle_\mu = \sum_{v \in \mathcal{V}} f(v)g(v)\mu(v)$. The set of functions $f : \mathcal{V} \rightarrow \mathbb{R}$ which are 0 at all but finitely many vertices is denoted by \mathcal{D}_K . Also introduce $\mathcal{D}_{\mathbb{A}, \mu} = \mathbb{A} \cap l^2(\mu)$.

The next proposition collects basic facts about the symmetric bilinear forms induced by the edge conductances $C(u, v)$. Closely related results using the smaller domain \mathcal{D}_K are in [4, p. 20] and [16].

Proposition 3.1. *Suppose $C : \mathcal{V} \times \mathcal{V} \rightarrow [0, \infty)$ satisfies $C(u, v) = C(v, u)$, with $C(u, v) > 0$ if and only if $[u, v] \in \mathcal{E}$. Given a vertex weight μ , the symmetric bilinear form*

$$B(f, g) = \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{u \sim v} C(u, v) (f(v) - f(u))(g(v) - g(u)), \quad f, g \in \mathcal{D}_{\mathbb{A}, \mu}, \quad (3.1)$$

has a nonnegative quadratic form $B(f, f)$, and satisfies

$$B(f, g) = \langle \Delta_\mu f, g \rangle_\mu = \langle f, \Delta_\mu g \rangle_\mu, \quad (3.2)$$

where

$$\Delta_\mu f(v) = \frac{1}{\mu(v)} \sum_{u \sim v} C(u, v)(f(v) - f(u)). \quad (3.3)$$

Proof. The nonnegativity of the quadratic form is immediate from the definition. Note that for any $f \in \mathbb{A}$ there are only finitely many vertices $v \in \mathcal{V}$ for which $f(v) - f(u)$ is nonzero if u is adjacent to v .

To identify the operator Δ_μ , start with

$$\begin{aligned} 2B(f, g) &= \sum_{v \in \mathcal{V}} g(v) \sum_{u \sim v} C(u, v)(f(v) - f(u)) \\ &\quad - \sum_{v \in \mathcal{V}} \left(\sum_{u \sim v} C(u, v)g(u)(f(v) - f(u)) \right) \end{aligned} \quad (3.4)$$

Suppose a graph edge e has vertices $v_1(e)$ and $v_2(e)$. The second sum over $v \in \mathcal{V}$ in (3.4) can be viewed as a sum over edges, with each edge contributing the terms $C(v_1, v_2)g(v_1)(f(v_2) - f(v_1))$ and $C(v_1, v_2)g(v_2)(f(v_1) - f(v_2))$. Using this observation to change the order of summation gives

$$\begin{aligned} &\sum_{v \in \mathcal{V}} \left(\sum_{u \sim v} C(u, v)g(u)(f(v) - f(u)) \right) \\ &= \sum_{e \in \mathcal{E}} C(v_1(e), v_2(e)) \left(g(v_1)(f(v_2) - f(v_1)) + g(v_2)(f(v_1) - f(v_2)) \right) \\ &= \sum_{u \in \mathcal{V}} g(u) \sum_{v \sim u} C(u, v)(f(v) - f(u)) \end{aligned}$$

Employing this identity in (3.4) gives

$$2B(f, g) = 2 \sum_v \mu(v)g(v) \left[\frac{1}{\mu(v)} \sum_{u \sim v} C(u, v)(f(v) - f(u)) \right]. \quad (3.5)$$

□

With respect to the standard basis consisting of functions $\delta_w : \mathcal{V} \rightarrow \mathbb{R}$ with $\delta_w(w) = 1$ and $\delta_w(v) = 0$ for $v \neq w$, the operators Δ_μ have the matrix representation

$$Q(v, w) = \begin{cases} \mu^{-1}(w) \sum_{u \sim w} C(u, w), & v = w \\ -\mu^{-1}(v)C(v, w), & v \sim w \\ 0, & \text{otherwise} \end{cases}, \quad v, w \in \mathcal{V}.$$

If v is fixed, then summing on w gives

$$\sum_w Q(v, w) = \frac{1}{\mu(v)} \sum_{u \sim v} C(u, v) - \frac{1}{\mu(v)} \sum_{u \sim v} C(u, v) = 0,$$

so $-Q(v, w)$ is a Q -matrix in the sense of Markov chains [17, p. 58].

In the Q - matrix formulation the matrix entries represent transition rates, so decreasing the vertex measure μ increases the rates. Consistent with the boundary value themes of this work an interesting choice is to take the vertex weight $\mu_0(v)$ to be half the sum of the lengths of the incident edges,

$$\mu_0(v) = \frac{1}{2} \sum_{u \sim v} R(u, v).$$

This choice makes the vertex measure consistent with the previously defined graph volume,

$$\mu_0(\mathcal{G}) = \sum_{e \in \mathcal{E}} R(e) = \text{vol}(\mathcal{G}).$$

If $\text{vol}(\mathcal{G}) < \infty$, then $l^2(\mu_0)$ will include all functions in \mathbb{A} .

With respect to this measure

$$\Delta_{\mu_0} f(v) = \mu_0^{-1}(v) \sum_{u \sim v} C(u, v)(f(v) - f(u)) = \frac{2}{\sum_{u \sim v} R(u, v)} \sum_{u \sim v} \frac{f(v) - f(u)}{R(u, v)}.$$

This operator resembles the symmetric second difference operator from numerical analysis. Like the classical Laplace operator in Euclidean space, Δ_{μ_0} exhibits quadratic scaling behavior. Suppose $v_0, v_1 \in \mathcal{V}$ have equal numbers of incident edges, and there is a bijection of vertex neighborhoods with $R(v_1, w_1) = \rho R(v_0, w_0)$ for $\rho > 0$, and $w_j \sim v_j$. If $f : \mathcal{V} \rightarrow \mathbb{R}$ satisfies $f(v_1) = f(v_0)$ and $f(w_1) = f(w_0)$, then

$$\Delta_{\mu_0} f(v_1) = \rho^{-2} \Delta_{\mu_0} f(v_0).$$

The vertex weight μ_0 is typically distinct from $\mu(v) = \sum_{u \sim v} C(u, v)$, a choice which appears in the study of discrete time Markov chains [6, p. 40], [17, p. 73], [18, p. 18] with transition probabilities $p(u, v) = \mu^{-1}(v)C(u, v)$ for $u \neq v$.

3.2 Continuity

The next result considers continuous extension of functions to $\overline{\mathcal{G}}$ when the quadratic form is finite.

Theorem 3.2. *Suppose \mathcal{G} is connected. Using the metric of (2.1), functions $f : \mathcal{V} \rightarrow \mathbb{R}$ with $B(f, f) < \infty$ are uniformly continuous on \mathcal{G} , and so f extends uniquely to a continuous function on $\overline{\mathcal{G}}$.*

Proof. If $v, w \in \mathcal{V}$ and $\gamma = (v = v_0, v_1, \dots, v_K = w)$ is any finite simple path from v to w , then the Cauchy-Schwarz inequality gives

$$\begin{aligned} |f(w) - f(v)|^2 &= \left| \sum_k [f(v_{k+1}) - f(v_k)] \frac{C^{1/2}(v_{k+1}, v_k)}{C^{1/2}(v_{k+1}, v_k)} \right|^2 \\ &\leq \sum_k [C(v_{k+1}, v_k) (f(v_{k+1}) - f(v_k))^2] \sum_k R(v_{k+1}, v_k) \\ &\leq 2B(f, f) \sum_k R(v_{k+1}, v_k). \end{aligned}$$

There is a simple path with $\sum_k R(v_{k+1}, v_k) \leq 2d(v, w)$, so

$$|f(w) - f(v)|^2 \leq 4B(f, f)d(v, w), \quad (3.6)$$

which shows f is uniformly continuous on \mathcal{G} . By [20, p. 149] f extends continuously to $\overline{\mathcal{G}}$. □

The bilinear form may be used to define a 'Sobolev style' Hilbert space $H^1(\mu)$ with inner product

$$\langle f, g \rangle_{\mu,1} = \sum_v f(v)g(v)\mu(v) + B(f, g).$$

Let $H_0^1(\mu)$ be the closure of \mathcal{D}_K in $H^1(\mu)$.

Lemma 3.3. *If \mathcal{G} is connected with finite diameter, then there is a constant C such that*

$$\sup_{v \in \mathcal{V}} |f(v)| \leq C \|f\|_{\mu,1}, \quad (3.7)$$

so a Cauchy sequence in $H^1(\mu)$ is a uniform Cauchy sequence. The functions f in the unit ball of H^1 are uniformly equicontinuous [20, p. 29].

Proof. Fixing a vertex v_0 , (3.6) gives

$$|f(v)| \leq |f(v_0)| + |f(v) - f(v_0)| \leq \|f\|_{\mu,1}/\sqrt{\mu(v_0)} + 2\|f\|_{\mu,1}\text{diam}(\mathcal{G})^{1/2},$$

which is (3.7). The uniform equicontinuity follows from (3.6). \square

Theorem 3.4. *Suppose \mathcal{G} is connected and has finite diameter. If $f \in H_0^1(\mu)$, then f has a unique continuous extension to $\overline{\mathcal{G}}$ which is zero at all points $x \in \overline{\mathcal{G}} \setminus \mathcal{G}$.*

Proof. Any function $f \in H_0^1(\mu)$ is the limit in $H^1(\mu)$ of a sequence f_n from \mathcal{D}_K . The functions f and f_n have unique continuous extensions to $\overline{\mathcal{G}}$ by Theorem 3.2. The extended functions f_n satisfying $f_n(x) = 0$ for all $x \in \overline{\mathcal{G}} \setminus \mathcal{G}$. By Lemma 3.3 the sequence f_n converges to f uniformly on \mathcal{G} , so the extensions f_n converge uniformly to the extension f on $\overline{\mathcal{G}}$. Thus $f(x) = 0$ for all $x \in \overline{\mathcal{G}} \setminus \mathcal{G}$. \square

4 Boundary value problems

4.1 The basic Laplacian

Let $S_{K,\mu}$ denote the operator Δ_μ on $l^2(\mu)$ with the domain \mathcal{D}_K .

Proposition 4.1. *The operator $S_{K,\mu}$ is symmetric and nonnegative on $l^2(\mu)$. The adjoint operator $S_{K,\mu}^*$ on $l^2(\mu)$ acts by*

$$(S_{K,\mu}^*h)(v) = \Delta_\mu h(v) = \frac{1}{\mu(v)} \sum_{u \sim v} C(u,v)(h(v) - h(u))$$

on the domain consisting of all $h \in l^2(\mu)$ for which $\Delta_\mu h \in l^2(\mu)$.

Proof. The symmetry and nonnegativity of $S_{K,\mu}$ are given by (3.2). Since $S_{K,\mu}$ is densely defined, $S_{K,\mu}^*$ is the operator whose graph is the set of pairs $(h, k) \in l^2(\mu) \oplus l^2(\mu)$ such that

$$\langle S_{K,\mu}f, h \rangle_\mu = \langle f, k \rangle_\mu$$

for all $f \in \mathcal{D}_K$. Suppose $f_v = \frac{1}{\mu(v)}\delta_v$. Then for any h in the domain of $S_{K,\mu}^*$,

$$k(v) = (S_{K,\mu}^*h)(v) = \langle S_{K,\mu}f_v, h \rangle_\mu = \sum_w \left[\sum_{u \sim w} C(u,w)(f_v(w) - f_v(u)) \right] h(w)$$

$$= \frac{1}{\mu(v)} \sum_{u \sim v} C(u, v)(h(v) - h(u)).$$

□

Proposition 4.1 provides a basic Laplace operator, the Friedrich's extension [14, pp. 322-326] of $S_{K,\mu}$, whose domain is a subset of $H_0^1(\mu)$ the closure of \mathcal{D}_K in $H^1(\mu)$. Let $\mathcal{L}_{K,\mu}$ denote the Friedrich's extension of $S_{K,\mu}$. Several features of $\mathcal{L}_{K,\mu}$ are implied by the condition $\mu(\mathcal{G}) < \infty$.

Proposition 4.2. *Suppose \mathcal{G} is connected, with finite diameter and infinitely many vertices. If $\mu(\mathcal{G}) < \infty$, $f \in \text{domain}(\mathcal{L}_K)$, and $\|f\|_\mu = 1$, then $\mathcal{L}_{K,\mu}$ has the strictly positive lower bound*

$$\langle \mathcal{L}_{K,\mu} f, f \rangle_\mu = B(f, f) \geq \frac{1}{4\mu(\mathcal{G})\text{diam}(\mathcal{G})}, \quad (4.1)$$

Proof. The Friedrich's extension $\mathcal{L}_{K,\mu}$ of the nonnegative symmetric operator $S_{K,\mu}$ has the same lower bound, so it suffices to consider functions $f \in \mathcal{D}_K$. Since $\|f\|_\mu = 1$ there must be some vertex v where $f^2(v) \geq \mu^{-1}(\mathcal{G})$. Since f has finite support, there is another vertex u with $f(u) = 0$. An application of (3.6) gives

$$\mu^{-1}(\mathcal{G}) \leq f^2(v) = [f(v) - f(u)]^2 \leq 4B(f, f)d(u, v).$$

□

Proposition 4.3. *Suppose \mathcal{G} is connected, $\overline{\mathcal{G}}$ is compact, and $\mu(\mathcal{G})$ is finite. Let $S_{1,\mu}$ be a symmetric extension of $S_{K,\mu}$ in $l^2(\mu)$ whose associated quadratic form is*

$$\langle S_{1,\mu} f, f \rangle_\mu = B(f, f).$$

Then the Friedrich's extension $\mathcal{L}_{1,\mu}$ of $S_{1,\mu}$ has compact resolvent.

Proof. The resolvent of $\mathcal{L}_{1,\mu}$ maps a bounded set in $l^2(\mu)$ into a bounded set in $H^1(\mu)$. Suppose f_n is a bounded sequence in $l^2(\mu)$, with $g_n = (\mathcal{L}_1 - \lambda I)^{-1} f_n$. By Lemma 3.3 and the Arzela-Ascoli Theorem [20, p. 169] the sequence g_n has a uniformly convergent subsequence, which converges in $l^2(\mu)$. □

4.2 Harmonic functions

When considering harmonic functions, it will be convenient to treat vertices of degree one (boundary vertices) as part of the boundary of $\overline{\mathcal{G}}$. For instance, this convention allows finite graphs with degree one vertices to have nonconstant harmonic functions. Let \mathcal{V}_{int} denote the set of interior vertices, that is vertices with degree at least 2. Say that a function $f : \mathcal{V} \rightarrow \mathbb{R}$ is *harmonic on \mathcal{G}* if

$$f(v) = \frac{1}{\sum_{u \sim v} C(u, v)} \sum_{u \sim v} C(u, v) f(u) \quad (4.2)$$

for all vertices $v \in \mathcal{V}_{int}$. Since the value $f(v)$ of a harmonic function is the weighted average of the values at the adjacent vertices, f has a minimum or maximum at some $v \in \mathcal{V}_{int}$ of a connected graph if and only if f is constant. The set of harmonic functions is independent of the vertex weight μ , but harmonic functions may be described as those for which

$$\Delta_\mu f(v) = \frac{1}{\mu(v)} \sum_{u \sim v} C(u, v) (f(v) - f(u)) = 0, \quad v \in \mathcal{V}_{int}.$$

Proposition 4.4. *Suppose \mathcal{G} is connected and $\overline{\mathcal{G}}$ is compact. If $\overline{\mathcal{G}} \setminus \mathcal{G} \neq \emptyset$, then 0 is not an eigenvalue of an operator $\mathcal{L}_{K, \mu}$.*

Proof. If 0 were an eigenvalue of $\mathcal{L}_{K, \mu}$, then there would be a corresponding eigenfunction f which is positive somewhere. By Theorem 3.4, f would extend continuously to $\overline{\mathcal{G}}$, with $f(x) = 0$ for all $x \in \overline{\mathcal{G}} \setminus \mathcal{G}$. Since $\overline{\mathcal{G}}$ is compact, f has a positive maximum at some $x \in \mathcal{G}$. Since (4.2) holds at all vertices, f is constant, but then $\overline{\mathcal{G}} \setminus \mathcal{G} \neq \emptyset$ implies $f = 0$. \square

Define $\partial\overline{\mathcal{G}}$ to be the complement of \mathcal{V}_{int} in $\overline{\mathcal{G}}$. The Dirichlet problem asks whether every continuous function $F : \partial\overline{\mathcal{G}} \rightarrow \mathbb{R}$ has a continuous extension $f : \overline{\mathcal{G}} \rightarrow \mathbb{R}$ which is harmonic on \mathcal{G} . The following example shows that the Dirichlet problem is not always solvable.

As shown in Figure A, construct a graph \mathcal{G} beginning with a sequence of vertices v_n and with edges (v_n, v_{n+1}) , $n = 0, 1, 2, \dots$. Take $R(v_n, v_{n+1}) = 2^{-n-1}$. For each of the vertices v_n introduce 2^n additional vertices $u_{n,k}$, and edges $(v_n, u_{n,k})$ of length 1. In this example, $\partial\overline{\mathcal{G}}$ consists of the boundary vertices $u_{n,k}$ together with the boundary point at $v = \lim_n v_n$. All of these points are isolated in $\partial\overline{\mathcal{G}}$.

Suppose the function F satisfies

$$F(u_{n,k}) = 0, \quad F(v) = 1.$$

A computation with (4.2) shows that any harmonic extension f which is continuous on $\overline{\mathcal{G}}$ must satisfy the contradictory requirements

$$\lim_{n \rightarrow \infty} f(v_n) = 1, \quad \lim_{n \rightarrow \infty} f(v_n) = 3/4.$$

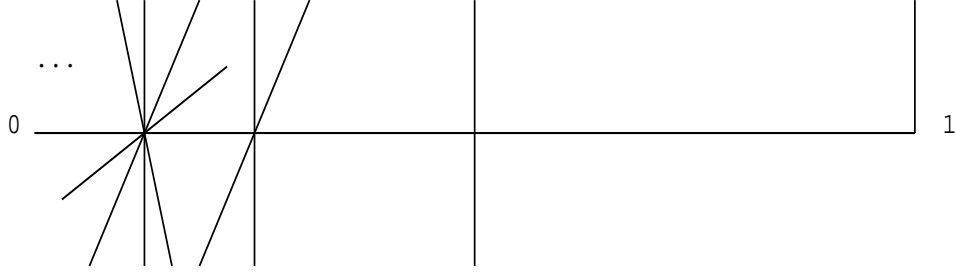


Figure A

The graph \mathcal{G} in this example is connected with finite diameter, $\overline{\mathcal{G}}$ is weakly connected, but $\overline{\mathcal{G}}$ is not compact. If $\overline{\mathcal{G}}$ is required to be compact, there is a positive result.

Theorem 4.5. *Suppose \mathcal{G} is connected, while $\overline{\mathcal{G}}$ is weakly connected and compact. Every continuous function $F : \partial\overline{\mathcal{G}} \rightarrow \mathbb{R}$ has a unique extension to a continuous function $f : \overline{\mathcal{G}} \rightarrow \mathbb{R}$ that is harmonic on \mathcal{G} .*

Proof. The proof breaks into two main parts. For the first part, assume that \mathcal{G} has no boundary vertices, so F is defined on $\overline{\mathcal{G}} \setminus \mathcal{G}$. The second part of the proof will extend this partial result to the full theorem.

The uniqueness is a standard consequence of the maximum principle since the difference of two solutions would be 0 on $\overline{\mathcal{G}} \setminus \mathcal{G}$. By the Tietze extension theorem [20, p. 179], F may be extended to a continuous function on $\overline{\mathcal{G}}$. By Theorem 2.12, for any $n > 0$ there is a $g_n \in \mathbb{A}$ with

$$\max_{x \in \overline{\mathcal{G}}} |F(x) - g_n(x)| \leq 1/n.$$

Pick vertex weights μ with $\mu(\mathcal{G}) < \infty$. Since $g_n \in \mathbb{A}$, the function $\Delta_\mu g_n$ has the value 0 except at a finite set of vertices. The compactness of $\overline{\mathcal{G}}$

means that \mathcal{G} has finite diameter. By Proposition 4.2 the operator $\mathcal{L}_{K,\mu}$ has a strictly positive lower bound, so the equation

$$-\mathcal{L}_\mu h_n = \Delta_\mu g_n, \quad h_n|_{\overline{\mathcal{G}} \setminus \mathcal{G}} = 0,$$

has a solution. The function $f_n = g_n + h_n$ is harmonic on \mathcal{V} , continuous on $\overline{\mathcal{G}}$, and satisfies

$$\max_{x \in \overline{\mathcal{G}} \setminus \mathcal{G}} |F(x) - f_n(x)| \leq 1/n.$$

Since $\{f_n\}$ converges uniformly on $\overline{\mathcal{G}} \setminus \mathcal{G}$, the maximum principle implies it is a uniformly Cauchy sequence on $\overline{\mathcal{G}}$, so there is a continuous limit f on $\overline{\mathcal{G}}$. At each $v \in \mathcal{V}$

$$\begin{aligned} f(v) &= \lim_{n \rightarrow \infty} f_n(v) = \lim_{n \rightarrow \infty} \frac{1}{\sum_{u \sim v} C(u, v)} \sum_{u \sim v} C(u, v) f_n(u) \\ &= \frac{1}{\sum_{u \sim v} C(u, v)} \sum_{u \sim v} C(u, v) f(u), \end{aligned}$$

so f is harmonic on \mathcal{G} . This completes the first main part of the proof.

The second main step involves extending the theorem to include boundary vertices. For each edge $e = [v_0, v_b]$ with boundary vertex v_b and edge length $r_e = C(v_0, v_b)^{-1}$, delete the vertex v_b and edge e from \mathcal{G} . For $n = 1, 2, 3, \dots$, add a sequence of new vertices v_n and edges $e_n = [v_n, v_{n+1}]$. Assume the edges e_n have length r_n with $\sum r_n = r_e$. The resulting graph \mathcal{G}_1 will have a completion $\overline{\mathcal{G}}_1$ containing a point $w = \lim_{n \rightarrow \infty} v_n$.

Suppose a continuous function $G : \overline{\mathcal{G}}_1 \setminus \mathcal{G}_1 \rightarrow \mathbb{R}$ is given, and is extended to the harmonic function g on \mathcal{G}_1 . With $C(v_n, v_{n+1}) = r_n^{-1}$ we have

$$\begin{aligned} g(v_n) &= \frac{1}{1/r_{n-1} + 1/r_n} \left[\frac{1}{r_{n-1}} g(v_{n-1}) + \frac{1}{r_n} g(v_{n+1}) \right] \\ &= \frac{r_n}{r_{n-1} + r_n} g(v_{n-1}) + \frac{r_{n-1}}{r_{n-1} + r_n} g(v_{n+1}), \end{aligned}$$

or

$$(g(v_n) - g(v_{n-1}))/r_{n-1} = (g(v_{n+1}) - g(v_n))/r_n,$$

and so g is a linear function of the distance along the path from v_0 to w .

At v_0 the fact that g is harmonic means

$$\sum_{u \sim v_0} C(u, v_0)(g(v_0) - g(u)) = 0.$$

The linearity of g along the path means that

$$C(v_1, v_0)(g(v_0) - g(v_1)) = r_0^{-1}(g(v_0) - g(v_1)) = r_e^{-1}(g(v_0) - g(w)).$$

Now define the function $f : \mathcal{G} \rightarrow \mathbb{R}$ by $f(v) = g(v)$ for $v \in \mathcal{G}_1 \cap \mathcal{G}$, and $f(v_b) = g(w)$. The function f is harmonic on \mathcal{G} , and extends to the continuous function $F : \overline{\mathcal{G}} \rightarrow \mathbb{R}$ where $F(x) = G(x)$ for $x \in \overline{\mathcal{G}} \setminus \mathcal{G}$, while $F(v_b) = G(w)$ for each boundary vertex v_b . □

4.3 Boundary conditions and operators

In this section absorbing and reflecting boundary conditions are used to construct distinct nonnegative self adjoint extensions of $S_{K,\mu}$. The constructed operators extend to semigroup generators which are positivity preserving contractions on $l^1(\mu)$.

Given a closed set $\Omega \subset \{\overline{\mathcal{G}} \setminus \mathcal{G}\}$, let \mathbb{A}_Ω denote the subalgebra of \mathbb{A} vanishing on Ω . Define the domain

$$\mathcal{D}_\Omega = \mathbb{A}_\Omega \cap l^2(\mu). \quad (4.3)$$

Let $S_{\Omega,\mu}$ denote the operator with domain \mathcal{D}_Ω acting on $l^2(\mu)$ by $S_\Omega f = \Delta_\mu f$.

By Proposition 3.1 the operator $S_{\Omega,\mu}$ is nonnegative and symmetric, with quadratic form $\langle S_{\Omega,\mu} f, f \rangle_\mu = B(f, f)$. Let $\mathcal{L}_{\Omega,\mu}$ denote the Friedrich's extension of $S_{\Omega,\mu}$, and note that the domain of $\mathcal{L}_{\Omega,\mu}$ is a subset of $H^1(\mu)$. A slight modification of the proof of Theorem 3.4 shows that every function f_j in the domain of $\mathcal{L}_{\Omega,\mu}$ extends continuously to $\overline{\mathcal{G}}$ with $f_j(x) = 0$ for $x \in \Omega$.

Given a compact set $\Omega \subset \overline{\mathcal{G}}$, let $N_R(\Omega) = \{z \in \overline{\mathcal{G}} \mid d(z, \Omega) \leq R\}$ be the set of points whose distance from Ω is at most R .

Theorem 4.6. *Suppose $\overline{\mathcal{G}}$ is connected, weakly connected, and compact. Assume that for $j = 1, 2$ the sets $\Omega(j) \subset \{\overline{\mathcal{G}} \setminus \mathcal{G}\}$ are compact, and that $\mu(N_R(\Omega(j))) < \infty$ for some $R > 0$. If $\Omega(1) \neq \Omega(2)$, then $\mathcal{L}_{\Omega(1),\mu}$ and $\mathcal{L}_{\Omega(2),\mu}$ have distinct domains.*

Proof. Reversing the roles of $\Omega(1)$ and $\Omega(2)$ if necessary, we may assume $\Omega(1) \not\subset \Omega(2)$. Let $\Omega(2)^c$ denote the complement of $\Omega(2)$. Find $x \in \Omega(1) \cap \Omega(2)^c$, and an r with $0 < r < R$ such that $N_r(x) \subset \Omega(2)^c$.

The sets $\{x\}$ and $V = \{y \in \overline{\mathcal{G}} \mid d(x, y) \geq r\}$ are disjoint and compact. Corollary 2.11 shows there is a function $f \in \mathbb{A}$ with $f(x) = 1$, while $f(z) = 0$

for all $z \in V$. Since $\mu(N_r(x)) < \infty$, the function f is in the domain of $\mathcal{L}_{\Omega(2),\mu}$, but f is not in the domain of $\mathcal{L}_{\Omega(1),\mu}$. \square

Since the operators $\mathcal{L} = \mathcal{L}_{\Omega,\mu}$ are nonnegative self adjoint on $l^2(\mu)$, the operators of the semigroup $\exp(-t\mathcal{L}_{\Omega,\mu})$ are $l^2(\mu)$ contractions for $t \geq 0$. For applications to probability this is not sufficient. Let $Quad(\mathcal{L})$ denote the domain of $\mathcal{L}^{1/2}$. The method of [4, p. 20] will show that the quadratic forms

$$Q(f) = \langle \mathcal{L}^{1/2}f, \mathcal{L}^{1/2}f \rangle_\mu, \quad f \in Quad(\mathcal{L})$$

associated to $\mathcal{L}_{\Omega,\mu}$ are Dirichlet forms.

There are two conditions to check. The first condition is that $f \in Quad(\mathcal{L})$ implies $|f| \in Quad(\mathcal{L})$ and $B(|f|, |f|) \leq B(f, f)$. Since the form is

$$B(f, f) = \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{u \sim v} C(u, v) (f(v) - f(u))^2,$$

the first condition holds for $f \in \mathcal{D}_\Omega$. If $f \in Quad(\mathcal{L})$ then there is a sequence $f_n \in \mathcal{D}_\Omega$ with

$$\langle f, f \rangle_\mu + B(f, f) = \lim_{n \rightarrow \infty} \langle f_n, f_n \rangle_\mu + B(f_n, f_n).$$

It follows that

$$\langle |f|, |f| \rangle_\mu + B(|f|, |f|) = \lim_{n \rightarrow \infty} \langle |f_n|, |f_n| \rangle_\mu + B(|f_n|, |f_n|),$$

and $B(|f|, |f|) \leq B(f, f)$.

The second condition is that if $f \in Quad(\mathcal{L})$ and $g \in l^2(\mu)$ with $|g(v)| \leq |f(v)|$ and $|g(v) - g(u)| \leq |f(v) - f(u)|$ for all $u, v \in \mathcal{V}$, then $g \in Quad(\mathcal{L})$ and $Q(g) \leq Q(f)$. This is even more transparent than the first condition.

Again quoting [4, p. 12-13], the following result is established.

Theorem 4.7. *For $t \geq 0$ the semigroups $\exp(-\mathcal{L}_{\Omega,\mu}t)$ on $l^2(\mu)$ are positivity preserving contractions on $l^p(\mu)$ for $1 \leq p \leq \infty$*

Arguing by analogy, one expects the operator $\mathcal{L} = \mathcal{L}_{\Omega,\mu}$ to exhibit 'absorption' at Ω and 'reflection' at Ω^c . This analogy may be tested by estimating the rate of decay of a probability density function initially supported near the 'reflecting' boundary.

Suppose $\overline{\mathcal{G}}$ is compact, connected and weakly connected. Using Theorem 2.6 and the subsequent remarks, let V be a clopen neighborhood of Ω , with $U = V^c$. The edge boundary of U will be the set $\partial_e U$ of edges $e = [u, v] \in \mathcal{G}$ with $u \in U$ and $v \in V$.

The edge boundary $\partial_e U$ is a finite set since by Theorem 2.4 there is a finite set of edges W such that any path from U to V contains an edge from W , and $\partial_e U \subset W$. Let 1_U be the indicator function of U , with $1_U(x) = 1$ for $x \in U$ and $1_U(x) = 0$ for $x \in V$. The function 1_U is in \mathbb{A} . If $\mu(V) < \infty$ then 1_U is in the domain of \mathcal{L} .

Suppose $p_0 : \mathcal{V} \rightarrow [0, \infty)$ is a bounded probability density with $p_0(v) \geq 0$ and $\sum_{\mathcal{G}} p_0(v)\mu(v) = 1$. For $t \geq 0$ define $p(t, v) = \exp(-t\mathcal{L})p_0(v)$ and denote the density integrated over U by $P_U(t) = \sum_{v \in U} p(t, v)\mu(v)$.

Using $p_0 \in l^2(\mu)$, we find that for $t > 0$,

$$\begin{aligned} \frac{d}{dt}P_U(t) &= \frac{d}{dt}\langle \exp(-t\mathcal{L})p_0, 1_U \rangle_\mu = -\langle \mathcal{L} \exp(-t\mathcal{L})p_0, 1_U \rangle_\mu \\ &= -\langle \exp(-t\mathcal{L})p_0, \mathcal{L}1_U \rangle_\mu = - \sum_{(u,v) \in \partial_e U} C(u, v)(p(t, u) - p(t, v)). \end{aligned}$$

Since the semigroup is positivity preserving and a contraction on $l^\infty(\mu)$,

$$\frac{d}{dt}P_U(t) \geq -\|p_0\|_\infty \sum_{(u,v) \in \partial_e U} C(u, v). \quad (4.4)$$

That is, the decay rate for $P_U(t)$ is controlled by what happens at $\partial_e U$, without regard to the rest of the boundary of U .

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